Least square approximation

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In the following four different least square methods are derived. The values to be approximated are gived by $D(j)$. The first two methods use a linear dependency on the approximating function, whereas the last two are on a more general form. The later are only approximation and require an iterative process.

1 Real values

The method can as an example be used for polynomial fit.

$$\frac{d}{da_k}\sum_j W(j)(\sum_i a_i f(i,j) - D(j))^2 = 0$$

$$\sum_j W(j)(\sum_i a_i f(i,j) - D(j))f(k,j) = 0$$

$$\sum_i a_i \sum_j W(j)f(i,j)f(k,j) = \sum_j W(j)f(k,j)D(j)$$

2 Complex values

Since the function f can be complex, the method can be used for FIR filter design.

$$\frac{d}{da_k}\sum_j W(j)|\sum_i a_i f(i,j) - D(j)|^2 = 0$$

$$\frac{d}{da_k}\sum_j W(j)(\sum_i a_i f(i,j) - D(j))(\sum_i a_i f^*(i,j) - D^*(j)) = 0$$

$$\sum_j W(j)f(k,j)(\sum_i a_i f^*(i,j) - D^*(j)) + W(j)f^*(k,j)(\sum_i a_i f(i,j) - D(j)) = 0$$

$$Re(\sum_j W(j)f(k,j)(\sum_i a_i f^*(i,j) - D^*(j))) = 0$$

$$Re\sum_j W(j)f(k,j)\sum_i a_i f^*(i,j) = Re\sum_j W(j)f(k,j)D^*(j)$$

$$\sum_i a_i \sum_j W(j)Re(f(k,j)f^*(i,j)) = \sum_j W(j)Re(f(k,j)D^*(j))$$

3 Complex values approximation

$$\frac{d}{da_k}\sum_j W(j)|H(j;\bar{\boldsymbol{a}}) - D(j)|^2 = 0$$

$$\sum_j W(j)Re((H^*(j;\bar{\boldsymbol{a}}) - D^*(j))\frac{\partial H(j;\bar{\boldsymbol{a}})}{\partial a_k}) = 0$$

H can be approcimated by:

$$H(j;\bar{\boldsymbol{a}}) \approx H_0(j) + \sum_i \frac{\partial H(j;\bar{\boldsymbol{a}})}{\partial a_i}\Delta a_i$$

$$\sum_j W(j)Re((H_0^*(j) + \sum_i \frac{\partial H^*(j;\bar{\boldsymbol{a}})}{\partial a_i}\Delta a_i - D^*(j))\frac{\partial H(j;\bar{\boldsymbol{a}})}{\partial a_k}) = 0$$

$$\sum_i \Delta a_i \sum_j W(j)Re(\frac{\partial H(j;\bar{\boldsymbol{a}})}{\partial a_k}\frac{\partial H^*(j;\bar{\boldsymbol{a}})}{\partial a_i}) = \sum_j W(j)Re(\frac{\partial H(j;\bar{\boldsymbol{a}})}{\partial a_k}(D^*(j) - H_0^*(j)))$$

4 Power values approximation

$$\frac{d}{da_k}\sum_j W(j)f(j;\bar{\boldsymbol{a}})^2 = 0$$

$$\sum_j W(j)f(j;\bar{\boldsymbol{a}})\frac{\partial f(j;\bar{\boldsymbol{a}})}{\partial a_k} = 0$$

f can be approcimated by:

$$f(j;\bar{\boldsymbol{a}}) \approx f_0(j) + \sum_i \frac{\partial f(j;\bar{\boldsymbol{a}})}{\partial a_i}\Delta a_i$$

$$\sum_j W(j)(f_0(j) + \sum_i \frac{\partial f(j;\bar{\boldsymbol{a}})}{\partial a_i}\Delta a_i)\frac{\partial f(j;\bar{\boldsymbol{a}})}{\partial a_k} = 0$$

$$\sum_j W(j)\sum_i \frac{\partial f(j;\bar{\boldsymbol{a}})}{\partial a_i}\Delta a_i \frac{\partial f(j;\bar{\boldsymbol{a}})}{\partial a_k} = -\sum_j W(j)f_0(j)\frac{\partial f(j;\bar{\boldsymbol{a}})}{\partial a_k}$$

$$\sum_i \Delta a_i \sum_j W(j)\frac{\partial f(j;\bar{\boldsymbol{a}})}{\partial a_i}\frac{\partial f(j;\bar{\boldsymbol{a}})}{\partial a_k} = -\sum_j W(j)f_0(j)\frac{\partial f(j;\bar{\boldsymbol{a}})}{\partial a_k}$$

Let:

$$f(j;\bar{\boldsymbol{a}}) = H(j;\bar{\boldsymbol{a}})H^*(j;\bar{\boldsymbol{a}}) - D(j)D(j)^*$$

Then:

$$\frac{\partial f(j;\bar{\boldsymbol{a}})}{\partial a_i} = H(j;\bar{\boldsymbol{a}})\frac{\partial H^*(j;\bar{\boldsymbol{a}})}{\partial a_k} + H^*(j;\bar{\boldsymbol{a}})\frac{\partial H(j;\bar{\boldsymbol{a}})}{\partial a_k} = 2Re(H^*(j;\bar{\boldsymbol{a}})\frac{\partial H(j;\bar{\boldsymbol{a}})}{\partial a_k})$$

If H can be written:

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{i=0}^{N} b_i z^{-i}}{1 + \sum_{i=1}^{M} a_i z^{-i}}$$

then:

$$\frac{\partial H(z;\bar{\boldsymbol{b}},\bar{\boldsymbol{a}})}{\partial c_k} = \begin{cases} \frac{1}{A(z)} z^{-k}, for\ c_k = b_k \\ \\ \frac{-H(z)}{A(z)} z^{-k}, for\ c_k = a_k \end{cases}$$

If H can be written:

$$H(z) = \frac{B(z)}{A(z)} = s\frac{\prod_{i=0}^{N} z^i - z_i}{\prod_{i=1}^{N} z^i - p_i}$$

where $z_i$and p_i are real, then:

$$\frac{\partial H(z;\mathrm{s},\bar{\boldsymbol{z}},\bar{\boldsymbol{p}})}{\partial c_k} = \begin{cases} \frac{H(z)}{s}, for\ c_k = s \\ \\ \frac{-H(z)}{z-z_k}, for\ c_k = z_k \\ \\ \frac{H(z)}{z-p_k}, for\ c_k = p_k \end{cases}$$