Signal Analysis Formulas

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Change log

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1. Document restarted. It is based on an older but similar document

11. august 2009

1. Formula for exponential averaging changed to give a better approximation for small values of N

21. november 2015

1. Formula for A-weighting corrected

Contents

1 The Discrete Fourier Transform **1**

1.1 Definition of the Discrete Fourier transform 1
1.2 Some rules for real signals 1
1.3 A few words about zoom 2

2 Baseband narrow band spectra and time functions **3**

2.1 Fourier spectrum 3
2.1.1 Calculation of magnitude 3
2.2 Autospectrum 3
2.2.1 Calculation of magnitude 4
2.3 Enhanced spectrum 4
2.3.1 Calculation of magnitude 4
2.4 Cross spectrum using spectrum averaging 4
2.4.1 Calculation of magnitude 5
2.5 Cross spectrum using signal enhancement 5
2.5.1 Calculation of magnitude 5
2.6 Frequency response 5
2.6.1 Calculation of magnitude 6
2.7 Impulse response 6
2.7.1 Calculation of magnitude 6
2.8 Coherence 6
2.9 Autocorrelation coefficient 6
2.9.1 Calculation of magnitude 7
2.10 Cross correlation coefficient 7
2.10.1 Calculation of magnitude 7

3 Zoom narrow band spectra and time functions **8**

3.1 Fourier spectrum 8
3.1.1 Calculation of magnitude 8
3.2 Autospectrum 8
3.3 Enhanced spectrum 8
3.3.1 Calculation of magnitude 9
3.4 Cross spectrum using spectrum averaging 9
3.5 Cross spectrum using signal enhancement 9
3.6 Frequency response 9
3.7 Impulse response 9
3.7.1 Calculation of magnitude 9
3.8 Coherence 9

3.9 Autocorrelation coefficient 10
3.9.1 Calculation of magnitude 10
3.10 Cross correlation coefficient 10
3.10.1 Calculation of magnitude 10

4 Equalization 11

5 Group delay calculations 12

6 Delay compensation of phase functions 14
6.1 Frequency domain functions 14
6.2 Time domain functions 14

7 Bow-tie correction 16
7.1 Autocorrelation 16
7.2 Cross correlation 16

8 Integration and differentiation of magnitude spectra 17
8.1 Integration 17
8.2 Differentiation 17

9 A-weighting of spectra 18

10 PSD and ESD for narrow bandspectra 19

11 Window definitions 20
11.1 Uniform 20
11.2 Hanning 20
11.3 Moriat 21
11.4 Hamming 21
11.5 Blackman - Harris 21
11.6 3. order Blackman - Harris 22
11.7 Hee - Low sidelobe 22
11.8 User defined 22
11.9 Kaiser-Bessel 23
11.10Flattop 23
11.11Transient 23
11.12Short hanning 24
11.13Exponential 24
11.14CosCos 24

12 Puretone detection 25
12.1 Hanning window 25
12.1.1 Case 1 26
12.1.2 Case 2 26
12.1.3 Case 3 26
12.2 Low Sidelobe window 26
12.2.1 Case 1 26
12.2.2 Case 2 26

12.2.3 Case 3 27
12.3 All other windows 27

13 Distortion measurements **28**
13.1 Finding the frequencies of the input signal 28
13.2 Finding the frequency of the harmonics and the sidebands . . 28
13.2.1 Harmonic Distortion 28
13.2.2 Difference Frequency Distortion 29
13.2.3 Intermodulation Distortion 29
13.3 Level of the input signal, harmonics and sidebands 29
13.4 Distortion calculation 29
13.4.1 Harmonic Distortion 30
13.4.2 Difference Frequency Distortion 30
13.4.3 Intermodulation Distortion 30

14 N'th Octave Synthesis **31**

15 Probability functions **33**
15.1 Probability density 33
15.2 Probability distribution 33
15.3 Mean 34
15.4 Variance 34
15.5 Standard deviation 34
15.6 Skewness 34
15.7 Kurtosis 35

16 Averaging **36**
16.1 Linear average 36
16.2 Exponential average 38

Chapter 1

The Discrete Fourier Transform

1.1 Definition of the Discrete Fourier transform

The definition of the Discrete Fourier transform (DFT) used in the following is given by:

$$X(k) = DFT_N\{x(n)\} = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn}, \qquad k = 0, ..., N-1$$

This definition is also used by the most common FFT implementations.
The Inverse Discrete Fourier transform (IDFT) is defined by:

$$x(n) = IDFT_N\{X(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi}{N}nk}$$

IDFT can be calculated from the DFT as follows:

$$x(n) = IDFT_N\{X(k)\} = \frac{1}{N}DFT_N\{X^*(k)\}^*$$

1.2 Some rules for real signals

For the real signal $x(n)$ and the corresponding spectrum $X(k)$ we have:

$$X(k) = X^*(-k)$$

where:

$$X(k) = DFT_N\{x(n)\}$$

and:

$$x(n) = IDFT_N\{X(k)\} = \frac{1}{N}DFT_N\{X(N-k)\} = Re(IDFT_N\{Z(k)\})$$

where:

$$Z(k) = \begin{cases} X(k) & \text{for } k = 0, N/2 \\ 2X(k) & \text{for } 1 \le k \le N/2 - 1 \\ 0 & \text{for } N/2 + 1 \le k \le N - 1 \end{cases}$$

The Hilbert transform of $x(n)$ is given by:

$$y(n) = Im(IDFT_N\{Z(k)\})$$

1.3 A few words about zoom

Zoom is a method for increasing the frequency resolution of the spectrum of a signal in a frequency band through a series of signal transformations.
The frequency band of interest must lie within the frequency range of the signal, that is:

$$\frac{1}{n_d} f_{max} \leq f_c \leq f_{max} - \frac{1}{n_d} f_{max}$$

where:
f_{max} is the frequency range of the signal.
f_c is the center frequency of the frequency band of interest.
n_d is the decimation factor corresponding to the increase in frequency resolution.

Frequency shift

$$y(n) = x(n)e^{-j2\pi f_c n} = x(n)(\cos(2\pi f_c n) - j\sin(2\pi f_c n))$$

Lowpass filtering

$$z(n) = h(n) * y(n) = h(n) * y_{re}(n) + jh(n) * y_{im}(n)$$

where:
h(n) is a lowpass filter. The cutoff frequency of the filter is half the frequency range of the frequency band of interest.

Decimation

$$v(n) = z(n_d n)$$

where:
v(n) is the complex time signal used in the spectrum calculations.

Chapter 2

Baseband narrow band spectra and time functions

2.1 Fourier spectrum

$$S(k) = \begin{cases} \frac{1}{N} X(k) & \text{for } k = 0 \\ \frac{\sqrt{2}}{N} X(k) & \text{for } 1 \leq k \leq k_{max} \end{cases}$$

where:

$$X(k) = DFT_N\{x(n)\}$$

$x(n)$: Time signal
$S(k)$: Fourier spectrum
k_{max}: Antialiasing filter cutoff linenumber

2.1.1 Calculation of magnitude

The magnitude is calculated as follows:

Rms: $G_1(k) = \sqrt{S_{re}^2(k) + S_{im}^2(k)}$

Pwr: $G_2(k) = G_1^2(k)$

Psd: $G_3(k) = \frac{1}{B_w} G_2(k)$

Esd: $G_4(k) = \frac{T_m}{\Delta f} G_2(k)$

The values of B_w, T_m are given in Chapter 10.

2.2 Autospectrum

$$G(k) = E\{|S(k)|^2\}$$

where:
$S(k)$: Fourier spectrum

$G(k)$: Autospectrum
$E\{\}$: Averaging operator

2.2.1 Calculation of magnitude

The magnitude has the following representations:

Rms: $G_1(k) = \sqrt{G(k)}$

Pwr: $G_2(k) = G(k)$

Psd: $G_3(k) = \frac{1}{B_w} G(k)$

Esd: $G_4(k) = \frac{T_m}{\Delta f} G(k)$

The values of B_w, T_m are given in Chapter 10.

2.3 Enhanced spectrum

$$S(k) = \begin{cases} \frac{1}{N} X(k) & \text{for } k = 0 \\ \frac{\sqrt{2}}{N} X(k) & \text{for } 1 \le k \le k_{max} \end{cases}$$

where:

$$X(k) = DFT_N\{x(n)\}$$

$x(n)$: Enhanced time signal
$S(k)$: Enhanced spectrum

2.3.1 Calculation of magnitude

See Fourier spectrum

2.4 Cross spectrum using spectrum averaging

$$G_{ab}(k) = G^*_{ba}(k) = E\{S^*_a(k) S_b(k)\}$$

where:
$S_a(k)$: Fourier spectrum of ChA
$S_b(k)$: Fourier spectrum of ChB
$G_{ab}(k)$: Cross spectrum
$E\{\}$: Averaging operator

2.4.1 Calculation of magnitude

The magnitude is calculated as follows:

Rms: $G_{1ab}(k) = \sqrt[4]{G^2_{ab_{re}}(k) + G^2_{ab_{im}}(k)}$

Pwr: $G_{2ab}(k) = G^2_{1ab}(k)$

Psd: $G_{3ab}(k) = \frac{1}{\sqrt{B_{w_a} B_{w_b}}} G_{2ab}(k)$

Esd: $G_{4ab}(k) = \frac{\sqrt{T_{m_a} T_{m_b}}}{\sqrt{\Delta f_a \Delta f_b}} G_{2ab}(k)$

The values of B_w, T_m are given in Chapter 10.

2.5 Cross spectrum using signal enhancement

$$G_{ab}(k) = G^*_{ba}(k) = S^*_a(k) S_b(k)$$

where:
$S_a(k)$: Enhanced spectrum of ChA
$S_b(k)$: Enhanced spectrum of ChB
$G_{ab}(k)$: Cross spectrum

2.5.1 Calculation of magnitude

See section about spectrum averaging

2.6 Frequency response

$$H_1(k) = \frac{G_{ab}(k)}{G_{aa}(k)}$$

$$H_2(k) = \frac{G_{bb}(k)}{G_{ba}(k)}$$

$$H_3(k) = \sqrt{\frac{G_{bb}(k)}{G_{aa}(k)}} e^{j\phi(k)}, \qquad \phi(k) = \angle G_{ab}(k)$$

where:
$G_{aa}(k)$: Autospectrum or magnitude of enhanced spectrum ChA in Pwr
$G_{bb}(k)$: Autospectrum or magnitude of enhanced spectrum ChB in Pwr
$G_{ab}(k)$: Cross spectrum

2.6.1 Calculation of magnitude

$$M(k) = \sqrt{H_{re}^2(k) + H_{im}^2(k)}$$

where:
$H(k)$: Corresponds to either $H_1(k)$, $H_2(k)$ or $H_3(k)$

2.7 Impulse response

$$h(n) = \begin{cases} g(n + N/2) & \text{for } 0 \leq n \leq N/2 - 1 \\ g(n - N/2) & \text{for } N/2 \leq n \leq N - 1 \end{cases}$$

where:

$$g(n) = \frac{1}{\Delta T} IDFT_N\{G(k)\}$$

$$G(k) = \begin{cases} H(k) & \text{for } k = 0 \\ 2H(k) & \text{for } 1 \leq k \leq k_{max} \\ 0 & \text{for } k_{max} + 1 \leq k \leq N - 1 \end{cases}$$

$H(k)$: Corresponds to either $H_1(k)$, $H_2(k)$ or $H_3(k)$
ΔT: is the sampling interval

2.7.1 Calculation of magnitude

$$M(n) = \sqrt{h_{re}^2(n) + h_{im}^2(n)}$$

2.8 Coherence

$$\gamma^2(k) = \frac{|G_{ab}(k)|^2}{G_{aa}(k)G_{bb}(k)}$$

where:
$G_{aa}(k)$: Autospectrum ChA in Pwr
$G_{bb}(k)$: Autospectrum ChB in Pwr
$G_{ab}(k)$: Cross spectrum
$\gamma^2(k)$: Coherence

2.9 Autocorrelation coefficient

$$r(n) = \frac{R(n)}{R(0)}$$

where:

$$R(n) = IDFT_N\{F(k)\}, \qquad \text{for } 0 \leq n \leq N/2 - 1$$

$$F(k) = \begin{cases} N^2 G(k) & \text{for } 0 \leq k \leq k_{max} \\ 0 & \text{for } k_{max} + 1 \leq k \leq N - 1 \end{cases}$$

$G(k)$: Autospectrum or magnitude of enhanced spectrum in Pwr

2.9.1 Calculation of magnitude

$$m(n) = \sqrt{r_{re}^2(n) + r_{im}^2(n)}$$

2.10 Cross correlation coefficient

$$r_{ab}(n) = \begin{cases} R_{ab}(n + N/2)/\sqrt{R_{aa}(0)R_{bb}(0)} & \text{for } 0 \leq n \leq N/2 - 1 \\ R_{ab}(n - N/2)/\sqrt{R_{aa}(0)R_{bb}(0)} & \text{for } N/2 \leq n \leq N - 1 \end{cases}$$

where:

$$R_{ab}(n) = IDFT_N\{F_{ab}(k)\}$$

$$F_{ab}(k) = \begin{cases} N^2 G_{ab}(k) & \text{for } 0 \leq k \leq k_{max} \\ 0 & \text{for } k_{max} + 1 \leq k \leq N - 1 \end{cases}$$

$R_{aa}(0)$: Sample 0 in Autocorrelation function ChA
$R_{bb}(0)$: Sample 0 in Autocorrelation function ChB
$G_{ab}(k)$: Cross spectrum

2.10.1 Calculation of magnitude

$$m_{ab}(k) = \sqrt{r_{ab_{re}}^2(k) + r_{ab_{im}}^2(k)}$$

Chapter 3

Zoom narrow band spectra and time functions

3.1 Fourier spectrum

$$S(k) = \begin{cases} \frac{\sqrt{2}}{N} X(N + k - k_{max}) & \text{for } 0 \leq k \leq k_{max} - 1 \\ \frac{\sqrt{2}}{N} X(k - k_{max}) & \text{for } k_{max} \leq k \leq 2k_{max} \end{cases}$$

where:

$$X(k) = DFT_N\{x(n)\}$$

$x(n)$: Time signal
$S(k)$: Fourier spectrum
k_{max}: Zoom filter cutoff line number

3.1.1 Calculation of magnitude

See Section 2.1.1

3.2 Autospectrum

See Section 2.2

3.3 Enhanced spectrum

$$S(k) = \begin{cases} \frac{\sqrt{2}}{N} X(N + k - k_{max}) & \text{for } 0 \leq k \leq k_{max} - 1 \\ \frac{\sqrt{2}}{N} X(k - k_{max}) & \text{for } k_{max} \leq k \leq 2k_{max} \end{cases}$$

where:

$$X(k) = DFT_N\{x(n)\}$$

$x(n)$: Enhanced time signal
$S(n)$: Enhanced spectrum

3.3.1 Calculation of magnitude

See Section 2.3.1

3.4 Cross spectrum using spectrum averaging

See Section 2.4

3.5 Cross spectrum using signal enhancement

See Section 2.5

3.6 Frequency response

See Section 2.6

3.7 Impulse response

$$h(n) = \begin{cases} g(n + N/2) & \text{for } 0 \le n \le N/2 - 1 \\ g(n - N/2) & \text{for } N/2 \le n \le N - 1 \end{cases}$$

where:

$$g(n) = \frac{1}{\Delta T} IDFT_N\{G(k)\}$$

$$G(k) = \begin{cases} H(k + k_{max}) & \text{for } 0 \le k \le k_{max} \\ 0 & \text{for } k_{max} + 1 \le k \le N - k_{max} - 1 \\ H(k - N + k_{max}) & \text{for } N - k_{max} \le k \le N - 1 \end{cases}$$

$H(k)$: Corresponds to either $H_1(k)$, $H_2(k)$ or $H_3(k)$
ΔT: is the sampling interval

3.7.1 Calculation of magnitude

See Section 2.7.1

3.8 Coherence

See Section 2.8

3.9 Autocorrelation coefficient

$$r(n) = \frac{R(n)}{R(0)}$$

where:

$$R(n) = IDFT_N\{F(k)\}, \qquad \text{for } 0 \leq n \leq N/2 - 1$$

$$F(k) = \begin{cases} N^2 G(k + k_{max}) & \text{for } 0 \leq k \leq k_{max} \\ 0 & \text{for } k_{max} + 1 \leq k \leq N - k_{max} - 1 \\ N^2 G(k - N + k_{max}) & \text{for } N - k_{max} \leq k \leq N - 1 \end{cases}$$

$G(k)$: Autospectrum or magnitude of enhanced spectrum in Pwr

3.9.1 Calculation of magnitude

See Section 2.9.1

3.10 Cross correlation coefficient

$$r_{ab}(n) = \begin{cases} R_{ab}(n + N/2)/\sqrt{R_{aa}(0)R_{bb}(0)} & \text{for } 0 \leq n \leq N/2 - 1 \\ R_{ab}(n - N/2)/\sqrt{R_{aa}(0)R_{bb}(0)} & \text{for } N/2 \leq n \leq N - 1 \end{cases}$$

where:

$$R_{ab}(n) = IDFT_N\{F_{ab}(k)\}$$

$$F_{ab}(k) = \begin{cases} N^2 G_{ab}(k + k_{max}) & \text{for } 0 \leq k \leq k_{max} \\ 0 & \text{for } k_{max} + 1 \leq k \leq N - k_{max} - 1 \\ N^2 G_{ab}(N - 1 - k) & \text{for } N - k_{max} \leq k \leq N - 1 \end{cases}$$

$R_{aa}(0)$: Sample 0 in Autocorrelation function ChA
$R_{bb}(0)$: Sample 0 in Autocorrelation function ChB
$G_{ab}(k)$: Cross spectrum

3.10.1 Calculation of magnitude

See Section 2.10.1

Chapter 4

Equalization

Equalization is mainly used to compensate a frequency respons or impulse response for errors caused by differences in the two measurement channels. Differences in the amplitude and phase characteristics can be eliminated by dividing the measured frequency response by a reference frequency response. The reference frequency response can be obtained by measuring the frequency response of the instrument, by using the same input directly on the two channels of the instrument. Equalization can also be used for compensation for other measurement errors, by using a suitable reference.

Equalization of the frequency response is calculated as follows:

$$H_e(n) = \frac{H(n)}{H_r(n)} = \frac{|H(n)|}{|H_r(n)|} e^{j(\phi(n)-\phi_r(n))}$$

where:
$H(n)$ is the measured frequency response.
$H_r(n)$ is the reference frequency response.

The equalized impulse response is calculated from the equalized frequency response.

Chapter 5

Group delay calculations

The group delay of a continuous complex signal is defined as:

$$g_d(x) = -\frac{1}{2\pi}\frac{d\phi(x)}{dx}$$

where:

$$\phi(x) = \arctan\frac{s_{im}(x)}{s_{re}(x)}$$

$\phi(x)$ is in radians.
x is the independent variable (eg. time (s) or frequency (Hz)).
$s(n)$ is the signal or spectrum.

For a discrete signal an often used approximation to the group delay is given by:

$$g_d(n) = \begin{cases} -\frac{1}{2\pi}\frac{\phi(n+1)-\phi(n)}{\Delta x} & \text{for } 0 \le n \le n_{max}-1 \\ -\frac{1}{2\pi}\frac{2\phi(n)-3\phi(n-1)+\phi(n-2)}{\Delta x} & \text{for } n = n_{max} \end{cases}$$

where:
Δx is the time or frequency resolution.

If the phase has discontinuities then the formula will lead to highly incorrect results at the discontinuities. Another method is therefore often preferable:

$$g_d(n) = \begin{cases} -\frac{1}{2\pi}\frac{\phi_{cont}(1)-\phi_{cont}(0)}{\Delta x} & \text{for } n = 0 \\ -\frac{1}{2\pi}\frac{\phi_{cont}(n+1)-\phi_{cont}(n-1)}{2\Delta x} & \text{for } 1 \le n \le n_{max}-1 \\ -\frac{1}{2\pi}\frac{\phi_{cont}(n_{max})-\phi_{cont}(n_{max}-1)}{\Delta x} & \text{for } n = n_{max} \end{cases}$$

where:

$$\phi_{cont}(n) = \begin{cases} \phi(0) & \text{for } n = 0 \\ \phi(n) - 2\pi N_{rotations}(n) & \text{for } 1 \le n \le n_{max} \end{cases}$$

and

$$N_{rotations}(n) = \text{round}(\frac{\phi(n) - \phi_{cont}(n-1)}{2\pi})$$

$\phi(n)$ is calculated the usual way using Arctan and taking into account the sign. ϕ must be in the range $-\pi \leq \phi < \pi$.

Chapter 6

Delay compensation of phase functions

When a measurement is delayed in time the effect in the frequency domain is a linear phase contribution to the frequency domain functions. Correspondingly will a frequency shift have a similar effect in the time domain.
It is possible to compensate for such a time delay (or frequency shift) by multiplying the complex spectrum (time signal) by a complex exponential with a linear increasing phase. If only the phase function is to be compensated a linear phase can be added to the phase function.

6.1 Frequency domain functions

The frequency domain functions (e.g. spectrum or frequency response) can be compensated for a time delay as follows:

$$S_c(n) = S(n)e^{j\delta n}$$

where:

$$\delta = 2\pi\delta_t\Delta f = 2\pi\delta_t\frac{f_s}{N}$$

δ_t is the time delay in the measurement.
Δf is the frequency resolution.
f_s is the sampling frequency.
N is the record length.

6.2 Time domain functions

The time domain functions (e.g. time signal or correlation function) can be compensated for a frequency shift as follows:

$$T_c(n) = T(n)e^{j\delta n}$$

where:

$$\delta = 2\pi\delta_f\Delta T = 2\pi\delta_f\frac{1}{f_s}$$

δ_f is the frequency shift in the measurement.
ΔT is the sampling interval.
f_s is the sampling frequency.

Chapter 7

Bow-tie correction

7.1 Autocorrelation

$$r_c(n) = r(n)/(1 - 2n/N), 0 \leq n \leq N/2 - 1$$

7.2 Cross correlation

$$r_{c_{ab}}(n) = \begin{cases} r_{ab}(n)N/(2n) & \text{for } 0 \leq n \leq N/2 - 1 \\ r_{ab}(n)/(-2n/N + 2) & \text{for } N/2 \leq n \leq N - 1 \end{cases}$$

Chapter 8

Integration and differentiation of magnitude spectra

8.1 Integration

The RMS-magnitude spectrum of the integral of a signal is given by:

$$\tilde{S}(n) = \frac{1}{n\Delta f} S(n)$$

where:
Δf is the frequency resolution.
$S(n)$ is the RMS-magnitude spectrum of the signal.
$\tilde{S}(n)$ is the RMS-magnitude spectrum of the integral of the signal.

8.2 Differentiation

The RMS-magnitude spectrum of the derivative of a signal is given by:

$$\tilde{S}(n) = n\Delta f S(n)$$

where:
Δf is the frequency resolution.
$S(n)$ is the RMS-magnitude spectrum of the signal.
$\tilde{S}(n)$ is the RMS-magnitude spectrum of the derivative of the signal.

Chapter 9

A-weighting of spectra

The magnitude of the frequency response of the A-weighting filter is given by:

$$H_A(f) = \frac{f_4^2 f^4}{(f^2 + f_1^2)\sqrt{(f^2 + f_2^2)(f^2 + f_3^2)}(f^2 + f_4^2)}$$

where:
$f_1 = 20.59899706$
$f_2 = 107.6526486$
$f_3 = 737.8622307$
$f_4 = 12194.21715$
The A-weighted RMS spectrum is given by:

$$S_A(f) = H_A(f)S(f)$$

where:
$S(f)$ is the unweighted RMS spectrum.
$S_A(f)$ is the A-weighted RMS spectrum.

When A-weighting N'th octave spectra this method is only valid when $N \geq 24$. The following steps must therefore be followed:

1. Calculate an N'th octave spectrum with $N \geq 24$.
2. A-weighting as described above
3. Calculate the M'th octave spectrum using the method described in Chapter 14.

Chapter 10

PSD and ESD for narrow bandspectra

The factors B_w and T_m in the PSD and ESD calculations, depend on the windowtype and the linespacing in the spectrum as follows:

$$B_w = \delta_w \Delta f$$

$$T_m = \frac{\delta_m}{\Delta f}$$

where:
δ_w, δ_m: Depends on the chosen window. See Chapter 11
Δf: Line spacing in the spectrum in Hz

Chapter 11

Window definitions

All windows in the following are scaled as follows:

$$\frac{1}{N}\sum_{n=0}^{N-1} w(n) = 1$$

The correction factors are given by:

$$\delta_w = \frac{1}{N}\sum_{n=0}^{N-1} w^2(n)$$

$$\delta_m = \frac{1}{max(w(n))}$$

11.1 Uniform

$$w(n) = 1 \quad \text{for } 0 \leq n \leq N-1$$

$$\delta_w = 1$$

$$\delta_m = 1$$

11.2 Hanning

$$w(n) = 1 - \cos(\frac{2\pi}{N}n) \quad \text{for } 0 \leq n \leq N-1$$

$$\delta_w = 1.5$$

$$\delta_m = 0.5$$

11.3 Moriat

The window has a maximum asymptotic roll-off in the frequency domain (20(2P+1)dB/decade). In the time domain the first 2P-1 derivatives are 0 at the window ends. Note that the window is identical to the Hanning window for P=1.

$$\begin{aligned} w(n) &= 1 + \sum_{i=1}^{P} \beta_{P_i} \cos(\frac{2\pi}{N} in) \\ &= \frac{1}{\delta_m} \cos^{2P}(\frac{\pi}{N}(n - \frac{N}{2})) \quad \text{for } 0 \leq n \leq N-1 \end{aligned}$$

where:

$$\beta_{P_i} = 2(-1)^i \frac{(P!)^2}{(P-i)!(P+i)!}$$

$$\delta_w = 1 + \frac{1}{2} \sum_{i=1}^{P} \beta_{P_i}^2$$

$$\delta_m = 1/(1 + \sum_{i=1}^{P} |\beta_{P_i}|)$$

11.4 Hamming

The highest sidelobe is minimized and is attenuated by 43.2 dB.

$$w(n) = 1 + \beta_1 \cos(\frac{2\pi}{N} n) \quad \text{for } 0 \leq n \leq N-1$$

where:
$\beta_1 = -0.857516$

$$\delta_w = 1.3677$$

$$\delta_m = 0.5384$$

11.5 Blackman - Harris

The highest sidelobe is minimized and is attenuated by 71.4 dB.

$$w(n) = 1 + \sum_{i=1}^{2} \beta_i \cos(\frac{2\pi}{N} in) \quad \text{for } 0 \leq n \leq N-1$$

where:
$\beta_1 = -1.17213698$
$\beta_2 = +0.18462471$

$$\delta_w = 1.7034$$

$$\delta_m = 0.4243$$

11.6 3. order Blackman - Harris

The highest sidelobe is minimized and is attenuated by 98.1 dB.

$$w(n) = 1 + \sum_{i=1}^{3} \beta_i \cos(\frac{2\pi}{N} in) \quad \text{for } 0 \leq n \leq N-1$$

where:
$\beta_1 = -1.34550893$
$\beta_2 = +0.37578780$
$\beta_3 = -0.02928336$

$$\delta_w = 1.9762$$

$$\delta_m = 0.3636$$

11.7 Hee - Low sidelobe

The highest sidelobe is minimized and is attenuated by 125.3 dB.

$$w(n) = 1 + \sum_{i=1}^{4} \beta_i \cos(\frac{2\pi}{N} in) \quad \text{for } 0 \leq n \leq N-1$$

where:
$\beta_1 = -1.45875575$
$\beta_2 = +0.54308792$
$\beta_3 = -0.08816750$
$\beta_4 = +0.00390256$

$$\delta_w = 2.2154$$

$$\delta_m = 0.32323$$

11.8 User defined

$$w(n) = 1 + \sum_{i=1}^{P} \beta_i \cos(\frac{2\pi}{N} in) \quad \text{for } 0 \leq n \leq N-1$$

where:
The number of coefficients P and the coefficients β_i are specified by the user.

$$\delta_w = 1 + \frac{1}{2} \sum_{i=1}^{P} \beta_i^2$$

$$\delta_m = 1/(1 + \sum_{i=1}^{P} |\beta_i|)$$

11.9 Kaiser-Bessel

$$w(n) = 1 + \sum_{i=1}^{3} \beta_i \cos(\frac{2\pi}{N} in) \quad \text{for } 0 \leq n \leq N-1$$

where:
$\beta_1 = -1.23741757$
$\beta_2 = +0.24343057$
$\beta_3 = -0.00304230$

$$\delta_w = 1.7952$$

$$\delta_m = 0.4026$$

11.10 Flattop

The window has a minimum attenuation of sidelobes of 93.0 dB and a maix-mum mainlobe ripple of 0.0085 db.

$$w(n) = 1 + \sum_{i=1}^{4} \beta_i \cos(\frac{2\pi}{N} in) \quad \text{for } 0 \leq n \leq N-1$$

where:
$\beta_1 = -1.93261719$
$\beta_2 = +1.28613281$
$\beta_3 = -0.38769531$
$\beta_4 = +0.03222656$

$$\delta_w = 3.77$$

$$\delta_m = 0.21558$$

11.11 Transient

$$w(n) = \begin{cases} 0 & \text{for } 0 \leq n \leq D-1 \\ \frac{N}{L} & \text{for } D \leq n \leq D+L-1 \\ 0 & \text{for } D+L \leq n \leq N-1 \end{cases}$$

$$\delta_w = \frac{N}{L}$$

$$\delta_m = \frac{L}{N}$$

11.12 Short hanning

$$w(n) = \begin{cases} 0 & \text{for } 0 \le n \le D-1 \\ \frac{N}{L}(1-\cos(\frac{2\pi}{L}(n-D))) & \text{for } D \le n \le D+L-1 \\ 0 & \text{for } D+L \le n \le N-1 \end{cases}$$

$$\delta_w = 1.5\frac{N}{L} \quad \text{for } L \ge 2$$

$$\delta_m = 0.5\frac{L}{N}$$

11.13 Exponential

$$w(n) = \begin{cases} 0 & \text{for } 0 \le n \le D-1 \\ Ae^{-(n-D)/L} & \text{for } D \le n \le N-1 \end{cases}$$

where:
D: Delay
L: Equivalent length

$$A = N\frac{1-e^{-1/L}}{1-e^{-(N-D)/L}}$$

$$\delta_w = \frac{A^2}{N}\frac{1-e^{-2(N-D)/L}}{1-e^{-2/L}}$$

$$\delta_m = \frac{1}{A}$$

11.14 CosCos

$$w(n) = \begin{cases} 0 & \text{for } 0 \le n \le D-1 \\ 0.5A(1-\cos(\frac{\pi}{L_1}(n-D+1))) & \text{for } D \le n \le D+L_1-1 \\ A & \text{for } D+L_1 \le n \le D+L-L_2-1 \\ 0.5A(1-\cos(\frac{\pi}{L_2}(n-L-D))) & \text{for } D+L-L_2 \le n \le D+L-1 \\ 0 & \text{for } D+L \le n \le N-1 \end{cases}$$

where:
D: Delay
L_1: Length of head $L_1 \ge 2$
L_2: Length of tail $L_2 \ge 2$
L: Total length

$$A = \frac{2N}{2L-L_1-L_2+2}$$

$$\delta_w = \frac{A^2}{4N}(4L-2.5(L_1+L_2)+4)$$

$$\delta_m = \frac{1}{A}$$

Chapter 12

Puretone detection

This chapter describes a method for finding the pure tones in a spectrum. The spectrum levels used in the formulas are the RMS magnitude spectrum levels of either the instantaneous-, auto- or enhanced spectrum.
The pure tones are found at each pure tone peak. With the Hanning- and Low Sidelobe window, interpolation is used to improve the resolution. For all other windows the resolution is given by the line spacing in the spectrum.
A spectrum has a pure tone peak at each spectrum line where all conditions in one of the following cases are satisfied:

1.
$$G(n-2) < G(n-1) < G(n)$$
$$G(n+2) < G(n+1) < G(n)$$
$$G(n-1) = G(n+1)$$

2.
$$G(n-3) < G(n-2) < G(n-1)$$
$$G(n+2) < G(n+1) < G(n)$$
$$G(n-1) = G(n)$$

3.
$$G(n-2) < G(n-1) < G(n)$$
$$G(n+2) < G(n+1) < G(n)$$
$$G(n-1) \neq G(n+1)$$

where:
$G(n)$ is the RMS magnitude spectrum level at line n if $G(n) \geq G_0$, otherwise $G(n) = G_0$. G_0 is a prescribed level.

12.1 Hanning window

The line number frequency (n_p) and the level (G_p) of the pure tone is given by:

12.1.1 Case 1

$$n_p = n$$

$$G_p = G(n)$$

12.1.2 Case 2

$$n_p = n - 0.5$$

$$G_p = 1.17809725G(n)$$

12.1.3 Case 3

$$n_p = n + \delta_n$$

$$G_p = (1 - \delta_n^2)\frac{\pi\delta_n}{\sin\pi\delta_n}G(n)$$

where:

$$\delta_n = \frac{2 - G}{1 + G}\mathrm{Sign}(n_2 - n)$$

$$G = \frac{G(n)}{G(n_2}$$

n_2:Line number of the second higher spectrum line in the peak

12.2 Low Sidelobe window

The line number frequency (n_p) and the level (G_p) of the pure tone is given by:

12.2.1 Case 1

$$n_p = n$$

$$G_p = G(n)$$

12.2.2 Case 2

$$n_p = n - 0.5$$

$$G_p = 1.08144089G(n)$$

12.2.3 Case 3

$$n_p = n + \delta_n$$

$$G_p = (1/(1 + \delta_n^2 \sum_{i=1}^{4} \frac{\beta_i}{\delta_n^2 - i^2}))\frac{\pi\delta_n}{\sin \pi\delta_n} G(n)$$

where:

$$\frac{-2.4220141388G^2 + 0.3406758174G + 4.0856487460}{0.7525573617G^2 + 2.2560634875G + 1.0} \mathrm{Sign}(n_2 - n)$$

$$G = \frac{G(n)}{G(n_2)}$$

β_i Window coefficients
n_2:Line number of the second higher spectrum line in the peak

12.3 All other windows

For all three cases the line number frequency (n_p) and the level (G_p) of the pure tone is given by:

$$n_p = n$$

$$G_p = G(n)$$

Chapter 13

Distortion measurements

This chapter describes the methods for calculating the following types of distortion: Harmonic distortion, Difference Frequency Distortion and Intermodulation Distortion. The calculations are based on the RMS magnitude spectrum levels of either the instantaneous-, auto- or enhanced spectrum.
For each type of distortion, formulas for both the total distortion and the n'th order distortion is given. Although the total distortion is defined over an infinite frequency range, it is in practice limited by the frequency span of the spectrum.
The spectrum levels used in the formulas are the levels of the harmonics and sidebands of the input signal(singletone for Harmonic Distortion, twotone for the Difference Frequency- and Intermodulation Distortion). The frequency(ies) of the input signal can either be given exact or approximately. In the later case the exact frequency(ies) of the input signal is found from the spectrum by using the methods for puretone detection. In both cases the levels are found using the methods for puretone detection.

13.1 Finding the frequencies of the input signal

When the frequency(ies) of the input signal is given approximately the exact frequency(ies) are found by searching for the maximum level in the spectrum within 5 spectral lines from the given line number frequency (ies) and using the interpolation methods for puretone detection.

13.2 Finding the frequency of the harmonics and the sidebands

13.2.1 Harmonic Distortion

The frequency of the harmonics is given by:

$$f_n = nf_1 \quad n \geq 1$$

where:
f_1 is the exact frequency of the input frequency.
n is the distortion order.

13.2.2 Difference Frequency Distortion

The frequency of the sidebands is givne by:

$$w(n) = \begin{cases} \frac{n}{2}(f_2 - f_1) & \text{if } n \text{ even and } n \geq 2 \\ \frac{n+1}{2}f_2 - \frac{n-1}{2}f_1 & \text{if } n \text{ odd and } n \geq 3 \\ \frac{n+1}{2}f_2 - \frac{n-1}{2}f_1 & \text{if } n \text{ odd and } n \leq -3 \end{cases}$$

where:
f_1 is the exact frequency of the lowest input frequency.
f_2 is the exact frequency of the highest input frequency.
n is the distortion order.

13.2.3 Intermodulation Distortion

The frequency of the sidebands is givne by:

$$w(n) = \begin{cases} f_2 + (n-1)f_1 & \text{if } n \geq 2 \\ f_2 + (n+1)f_1) & \text{if } n \leq -2 \end{cases}$$

where:
f_1 is the exact frequency of the lowest input frequency.
f_2 is the exact frequency of the highest input frequency.
n is the distortion order.

13.3 Level of the input signal, harmonics and sidebands

The levels of the input signal (V_{f_1} and V_{f_2}) and the levels of the harmonics and sidebands (v_n) are calculated using the methods for puretone detection.

13.4 Distortion calculation

In the case where no harmonics or sidebands are present within the frequency span of the spectrum, the total distortion is 0 for all three types of distortion. The distortion of order 1 is always 1 for all three types of distortion.

13.4.1 Harmonic Distortion

The Total Harmonic Distortion is given by:

$$THD = \frac{\sqrt{\sum_{i=2}^{N} v_i^2}}{\sqrt{\sum_{i=1}^{N} v_i^2}}$$

The Harmonic Distortion of order n is given by:

$$HD_n = \frac{v_n}{\sqrt{\sum_{i=1}^{N} v_i^2}}$$

The upper limit N in the summation is given by the frequency span.

13.4.2 Difference Frequency Distortion

The Total Difference Frequency Distortion is given by:

$$TDFD = \frac{\sqrt{\sum_{i=2}^{N} v_i^2 + \sum_{j=3}^{M} (v_{+j} + v_{-j})^2}}{V_{f_1} + V_{f_2}}$$

The first summation is over the sidebands of even oder and the second over the side- bands of odd order.
The upper limits N and M in the summations are given by the frequency span. If the maximum distortion order is 2 then the second summation will vanish.

The Difference Frequency Distortion of order n is given by:

$$DFD_n = \begin{cases} \frac{v_n}{V_{f_1}+V_{f_2}} & \text{if } n \text{ even} \\ \frac{v_{+n}+v_{-n}}{V_{f_1}+V_{f_2}} & \text{if } n \text{ odd} \end{cases}$$

13.4.3 Intermodulation Distortion

The Total Intermodulation Distortion is given by:

$$TIMD = \frac{\sqrt{\sum_{i=2}^{N} (v_{+i} + v_{-i})^2}}{V_{f_2}}$$

The upper limit N in the summation is given by the frequency span.

The Intermodulation Distortion of order n is given by:

$$IMD_n = \frac{v_{+n} + v_{-n}}{V_{f_2}}$$

Chapter 14

N'th Octave Synthesis

N'th Octave Synthesis is a method for calculating an N'th octave spectrum based on a constant bandwidth spectrum.
An N'th octave spectrum is a constant percentage bandwidth spectrum, that is, a spectrum with a fixed ratio between the bandwidth and the center frequency of each band. The lower, upper and center frequencies of the p'th frequency band are given by:

N even:

$$f_{N_l}(p) = 10^{\frac{3p}{10N}}$$

$$f_{N_c}(p) = 10^{\frac{3p}{10N}} 10^{\frac{3}{20N}}$$

$$f_{N_u}(p) = 10^{\frac{3p}{10N}} 10^{\frac{6}{20N}}$$

N odd:

$$f_{N_l}(p) = 10^{\frac{3p}{10N}} 10^{\frac{-3}{20N}}$$

$$f_{N_c}(p) = 10^{\frac{3p}{10N}}$$

$$f_{N_u}(p) = 10^{\frac{3p}{10N}} 10^{\frac{3}{20N}}$$

As seen from the above, the cutoff frequencies are calculated on a decade basis rather than an octave basis in contrast with the name of the spectrum.
A PSD constant bandwidth spectrum (eg. a DFT spectrum) is converted to an N'th octave spectrum by first regarding the constant bandwidth spectrum as a stepwise constant function as follows:

$$\tilde{S}(f) = \tilde{S}(n\delta) = S(n), \qquad -\frac{1}{2}\Delta f \le \delta \le \frac{1}{2}\Delta f$$

where:
Δf is the frequency resolution of the constant bandwidth spectrum.
$S(n)$ is the PSD constant bandwidth spectrum.
$\tilde{S}(f)$ is the stepwise constant spectrum.

The N'th octave spectrum is then found by integrating $\tilde{S}(f)$ over each frequency band:

$$G_N(p) = \int_{f_{N_l}(p)}^{f_{N_u}(p)} \tilde{S}(f) df$$

where:
$G_N(p)$ is the power of the p'th frequency band of the N'th octave spectrum.

The PSD and BSD spectra are calculated as follows:

$$G_{N_{PSD}}(p) = \frac{1}{B_N(p)} G_N(p)$$

$$G_{N_{ESD}}(p) = \frac{T_N(p)}{B_N(p)} G_N(p)$$

where:
$B_N(p)$ is the bandwidth of the p'th frequency band of the N'th octave spectrum.
$T_N(p)$ is the measurement time of the p'th frequency band of the N'th octave spectrum.

From an N'th octave spectrum it is possible to calculate an M'th octave spectrum if N is divisible by M:

N/M even:

$$G_M(p) = \sum_{k=-k_1}^{k_1-1} G_N(\frac{N}{M}p + k), \qquad k_1 = \frac{1}{2}\frac{N}{M}$$

N/M odd:

$$G_M(p) = \sum_{k=-k_1}^{k_1} G_N(\frac{N}{M}p + k), \qquad k_1 = \frac{1}{2}(\frac{N}{M} - 1)$$

Chapter 15

Probability functions

15.1 Probability density

The probability density function is defined by:

$$f(n) = \frac{\text{Number of samples with a level in the interval}[l_n..l_{n+1}]}{\text{Total number of samples}}$$

The x-coordinate of the n'th interval is given by:

$$x(n) = (l_n + l_{n+1})/2 = \alpha n + \beta$$

where:
α and β are scaling constants.

From the definition it follows that:

$$\sum_{n=0}^{P-1} f(n) = 1$$

where:
P is the number of intervals.

Probability density functions with an interval width of αK are given by:

$$f_K(n) = \sum_{k=0}^{K-1} f(l+k), \qquad \left\{ \begin{array}{l} l = 0, K, ..., P-K \\ n = 0, 1, ..., P/K - 1 \end{array} \right.$$

15.2 Probability distribution

The probability distribution is given by:

$$F(n) = \sum_{l=0}^{n} f(l), \qquad n = 0, 1, ..., P-1$$

Probability distribution functions with an interval width of αK are given by:

$$F_K(n) \;\; = \;\; F((n+1)K - 1)$$

$$= \sum_{l=0}^{(n+1)K-1} f(l)$$

$$= \sum_{l=0}^{n} f_K(l), \quad n = 0, 1, ..., P/K - 1$$

15.3 Mean

The mean is given by:

$$\mu = \sum_{n=0}^{P-1} x(n) f(n) = \alpha\tilde{\mu} + \beta$$

where:

$$\tilde{\mu} = \sum_{n=0}^{P-1} n f(n)$$

15.4 Variance

The variance is given by:

$$\vartheta = \sum_{n=0}^{P-1} (x(n) - \mu)^2 f(n) = \alpha^2(\tilde{\vartheta} - \tilde{\mu}^2)$$

where:

$$\tilde{\vartheta} = \sum_{n=0}^{P-1} n^2 f(n)$$

15.5 Standard deviation

The standard deviation is given by:

$$\sigma = \sqrt{\vartheta}$$

15.6 Skewness

The skewness is given by:

$$\zeta = \sum_{n=0}^{P-1} (x(n) - \mu)^3 f(n) = \alpha^3(\tilde{\zeta} - 3\tilde{\mu}\tilde{\vartheta} + 2\tilde{\mu}^3)$$

where:

$$\tilde{\zeta} = \sum_{n=0}^{P-1} n^3 f(n)$$

15.7 Kurtosis

The kurtosis is given by:

$$\begin{aligned}\kappa &= \sum_{n=0}^{P-1} (x(n) - \mu)^4 f(n) \\ &= \alpha^4(\tilde{\kappa} - 4\tilde{\mu}\tilde{\zeta} + 6\tilde{\mu}^2\tilde{\vartheta} - 3\tilde{\mu}^4)\end{aligned}$$

where:

$$\tilde{\kappa} = \sum_{n=0}^{P-1} n^4 f(n)$$

Chapter 16

Averaging

Averaging is used to improve the accuracy of the estimation of DC and RMS levels. The only distinction between the procedures for estimation of DC and RMS levels is that the DC averaging is based on the signal sample values whereas the RMS averaging is based on the squared sample values and taking the squareroot of the result.
However, squaring a signal will double the cut off frequency, but since the signal is sampled there is a potential risk for aliasing. This is important when using the formulas for the averaging time to compute the ripple of the detected signal. In the following no distinction is made between DC and RMS calculation.
Averaging is a low-pass filtering of the signal by either an FIR or IIR filter with a frequency response of unity at zero frequency. The average is called a linear average when an FIR filter is used and an exponential average when an IIR filter is used.
The linear average is traditionally an average with an FIR filter with filter coefficients all having the same value. In the following the linear average does not have this limitation.
The exponential average is traditionally an average with a first order IIR filter. In the following only this case is considered.
Usually only some of the output samples are used and the linear averaging is greately simplified since only the samples of interest need to be calculated. For exponential averaging, however, all samples are used in the IIR filter feed back loop.
When the output sampling interval for linear averaging is shorter than the FIR filter impulse response length, the average is called a running average.

16.1 Linear average

In general a linear average can be written:

$$v_{avg}(n) = \frac{1}{A} \sum_{k=0}^{N-1} h(k) x(n + N - 1 + k)$$

$$A = \sum_{k=0}^{N-1} h(k)$$

where:
$v_{avg}(n)$ is the average output.
$h(k)$ is the averaging filter impulse response.
N is the length of the filter.
$x(k)$ is the input.

If $h(n) = h(N-1-n)$, that is, $h(n)$ is symmetrical then:

$$v_{avg}(n) = \frac{1}{A}\sum_{k=0}^{N-1} h(k)x(n+k)$$

and a single average is just the weighted sum of input samples given by:

$$v_{avg} = \frac{1}{A}\sum_{k=0}^{N-1} h(k)x(k)$$

Flat weighting

When $h(n) = 1$ for all n then the average is given by:

$$v_{avg}(n) = \frac{1}{N}\sum_{k=0}^{N-1} x(n+k)$$

Now, if the output of the average filter is sampled at a rate of N/P then the average can be written:

$$\begin{aligned} v_{avg}(\frac{N}{P}m) &= \frac{1}{N}\sum_{k=0}^{N-1} x(\frac{N}{P}m+k) \\ &= \frac{1}{N}\sum_{l=0}^{P-1}\sum_{k=0}^{N/P-1} x(\frac{N}{P}(m+l)+k) \\ &= \frac{1}{N}\sum_{l=0}^{P-1} x^*(m+l) \end{aligned}$$

$$x^*(m) = \sum_{k=0}^{N/P-1} x(\frac{N}{P}m+k)$$

or alternatively:

$$y_{avg}(m+1) = y_{avg}(m) + x^*(m+P) - x^*(m)$$

$$v_{avg}(\frac{N}{P}m) = \frac{1}{N}y_{avg}(m)$$

As seen from the above a flat weighted linear average with a sampling interval of $\frac{N}{P}$ can be computed recursively from a series of linear averages with length $\frac{N}{P}$.

Averaging time

The averaging time of a linear average with a weighting function $h(n)$ of length N is:

$$T_A = N\Delta T$$

where ΔT is the input sampling interval.

16.2 Exponential average

In general an exponential average is given by:

$$y_{avg}(n) = (1 - \frac{2}{N+1})y_{avg}(n-1) + \frac{2}{N+1}x(n)$$

The formula is only valid for $N \geq 1$. N should be set to 1 for no averaging.

Averaging time

The decay of the averaging is given by:

$$y_{avg}(n) = (1 - \frac{2}{N+1})^n \approx e^{-\frac{2}{N}n}$$

This corresponds to an RC network with a time constant of $\tau=$ RC, this means that $\frac{N}{2}\Delta T =$ RC. Where ΔT is the sampling interval
The effective averaging time for such an RC network is given in R.B. Randall, Frequency Analysis, p. 94:

$$T_A = 2RC$$

giving:

$$T_A = N\Delta T$$

where ΔT is the input sampling interval.