Sine and Sine Sweep measurements

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Chapter 1

Introduction

The frequency response of a system can be measured at a single frequency by applying a sine wave with a known amplitude to the system and recording the amplitude and phase of the output relative to the input as shown in Figure 1.1.

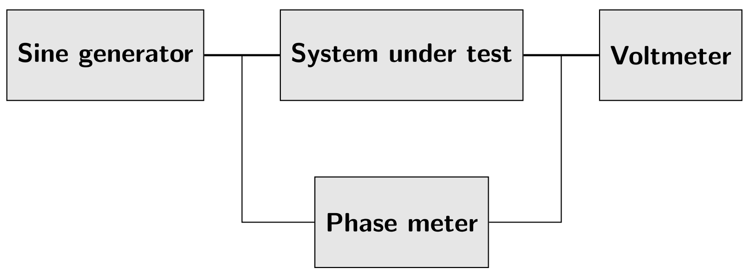

Figure 1.1: Basic frequency response measurement set-up

If the frequency of the generator is slowly changed, it is possible to obtain the frequency response over a frequency range. However, if the frequency of the generator is changed too fast the frequency response becomes incorrect. Figure 1.2, 1.3 and 1.4 shows the effect of sweeping a sine wave through a resonance at different sweep rates. As a rule of thumb the sweep rate S must be limited by:

$$S < (\Delta f)^2$$

where Δf is the bandwidth of the resonance.
Even if the sweep rate is low, the measurement becomes incorrect in situations where the S/N ratio is relatively poor. Therefore the heterodyne principle is often used as described in the next chapter.
Another widely used approach is the dual channel FFT method. However, the heterodyne method gives the possibility for overlapping data collection and signal processing and requires considerably less memory than pure FFT analysis.

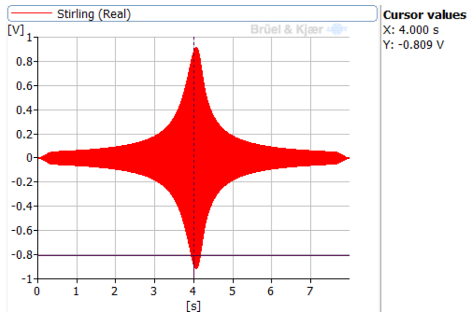

Figure 1.2: f_c = 3 kHz, Δf= 10 Hz, S= 25 Hz/s

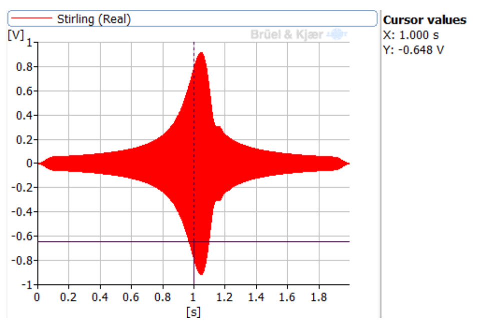

Figure 1.3: f_c = 3 kHz, Δf= 10 Hz, S= 100 Hz/s

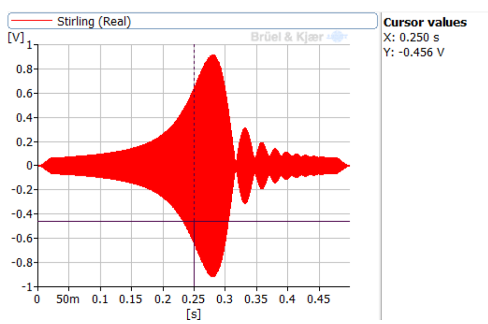

Figure 1.4: f_c = 3 kHz, Δf= 10 Hz, S= 400 Hz/s

Chapter 2

Heterodyne method

One of the advantages of the heterodyne method is its capability of improving the S/N ratio by the use of a tracking filter as shown in Figure 2.1. The method can be used for fixed frequency as well as for sine sweep analysis.

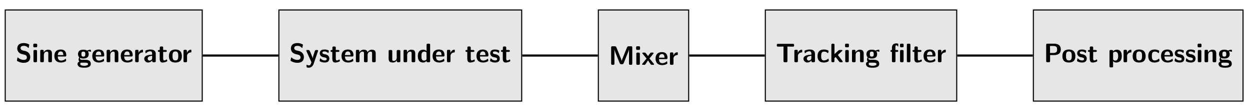

Figure 2.1: Heterodyne measurement set-up

In the mixer the output from the system under test is split into two signals where one is multiplied by a cosine version of the generator signal and the other by a sine version, see Figure 2.2.

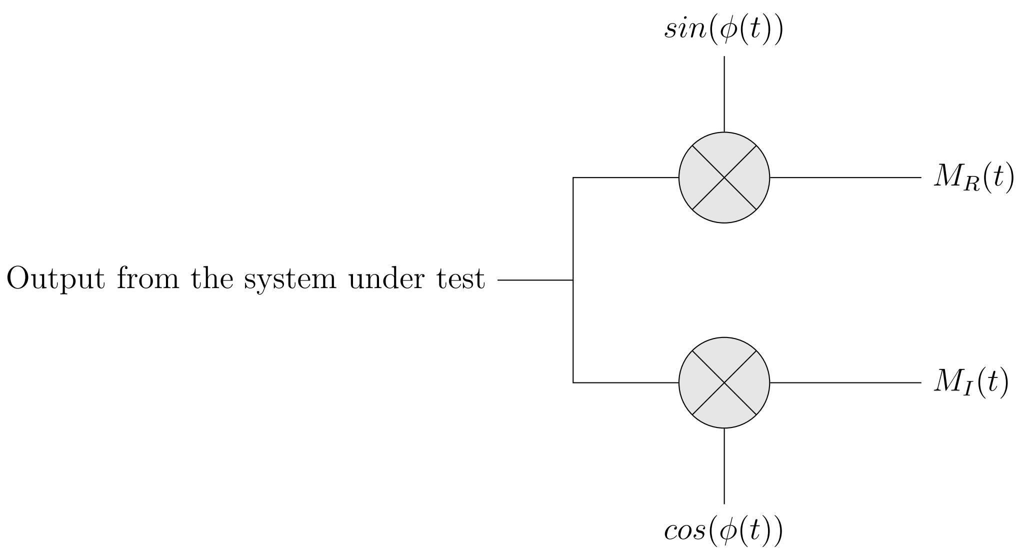

Figure 2.2: Heterodyne mixer. The generator signal is $sin(\phi(t))$

2.1 Fixed frequency analysis

If the generator signal is a sine wave of frequency ω_0, then the output of the SUT is also a sine wave of frequency ω_0, but with a different amplitude an phase:

$$v_{sut}(t) = |H(\omega_0)|sin(\omega_0 t + \phi_0), \qquad \phi_0 = \angle H(\omega_0)$$

The output of the mixer is:

$$M_R(t) = |H(\omega_0)|sin(\omega_0 t + \phi_0)sin(\omega_0 t) = 0.5|H(\omega_0)|(cos(\phi_0) - cos(2\omega_0 t + \phi_0))$$

$$M_I(t) = |H(\omega_0)|sin(\omega_0 t + \phi_0)cos(\omega_0 t) = 0.5|H(\omega_0)|(sin(\phi_0) + sin(2\omega_0 t + \phi_0))$$

The tracking filter consists of two low-pass filters each removing the component at the double frequency. The output of the tracking filter is:

$$T_R = 0.5|H(\omega_0)|cos(\phi_0)$$

$$T_I = 0.5|H(\omega_0)|sin(\phi_0)$$

The magnitude and phase of $H(\omega_0)$ is then given by:

$$|H(\omega_0)| = 2\sqrt{T_R^2 + T_I^2}$$

$$\phi_0 = \tan^{-1}(\frac{T_I}{T_R})$$

Apart from removing the component at the double frequency the low-pass filter also removes noise and thereby improves the S/N ratio of H. Moreover the low-pass filter may also remove distortion in case of non linear systems.

2.2 Sine sweep analysis

If the sweep rate is low (in the sense explained in Chapter 1) then the scheme outlined in the previous section can be used directly for sweep measurements, with ω_0 changing with time. Note that the low-pass filter cut-off frequency must be less than 2ω for all ω during the sweep. However, using a too low cut-off frequency will deteriorate the frequency response. As a rule of thumb the cut-off frequency f_{cut} is limited by:

$$f_{cut} > \frac{S}{\Delta f} \approx ST$$

where Δf is the bandwidth of the narrowest resonance in the frequency response and T is the length of the impulse response.
If the sweep rate is high the output of the system under test is not given by the $v_{sut}(t)$ above. The next sections describes how to overcome this problem.

2.2.1 Linear sweep

In this case the generator frequency is increasing proportionally with time:

$$\omega(t) = 2\pi St + \omega_{st}$$

$$\phi(t) = \pi St^2 + \omega_{st}t$$

where ω_{st} is the start frequency.
The tracking filter can be implemented as a FIR filter with a fixed f_{cut} and has four functions:

1. Removing the frequency component at the double frequency.
2. Removing noise.
3. Acting as an anti-aliasing filter enabling sampling reduction.
4. Removing distortion.

If the sweep rate is low no further processing is necessary, but for higher sweep rates the frequency response so far obtained is incorrect. Appendix A gives the details of the required processing. It can be summarized as follows:

1. Apply an IFFT to the false frequency response setting the negative frequencies to zero.
2. Multiply the result (the false impulse response) by a complex sine sweep in order to obtain the correct impulse response (real part).
3. Apply an FFT to the impulse response in order to obtain the correct frequency response.

Note that the size of the IFFT and FFT is much smaller than the sweep time due to the sampling reduction.

Example

Figure 2.3 shows the response of a system having three resonances at 1 kHz, 2.5 kHz and 4 kHz (B = 10 Hz), the sweep rate being 1 kHz/s.

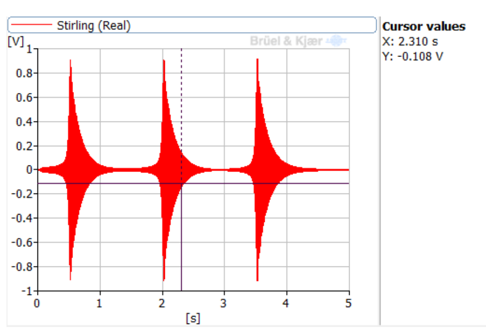

Figure 2.3: Linear sweep response

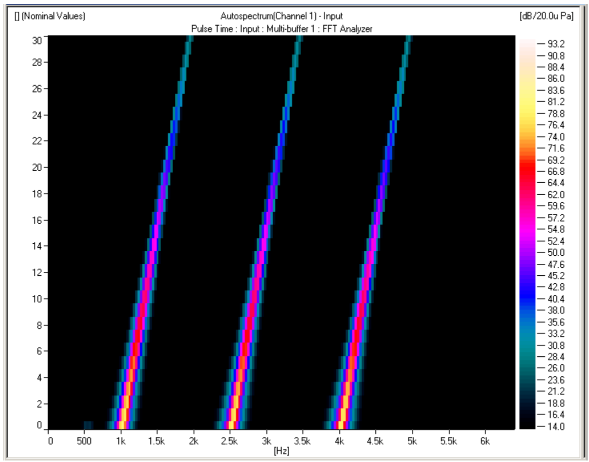

Figure 2.4: Contour plot of the false impulse response

Calculating the false impulse response and analysing it using short time FFT, a contour plot can be obtained showing the effect of the high sweep rate on the impulse response, Figure 2.4.
Note that the contour plot contains three parallel strait lines. Intuitively it makes sense that the impulse response can be corrected by multiplying by a sweep, since such an operation will shift the frequencies in the false impulse response linearly with time and thereby transforming the contour plot into Figure 2.5.

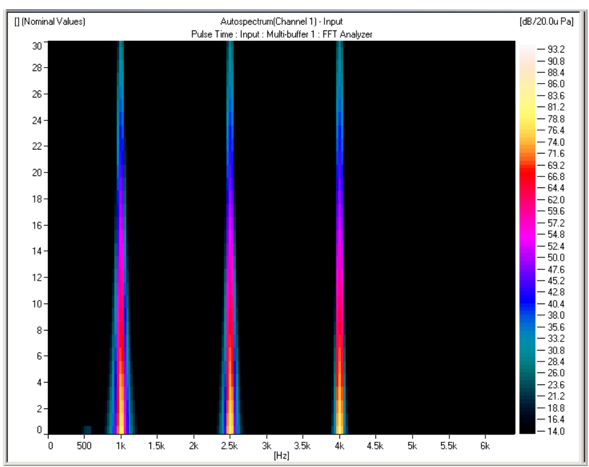

Figure 2.5: Contour plot of the corrected impulse response

2.2.2 Exponential sweep

Exponential sweep is traditionally called logarithmic sweep since it can be used for measuring the frequency response on a logarithmic frequency axis, but the generator frequency is increasing exponentially with time and is thus better described as an exponential sweep:

$$\omega(t) = \omega_{st} e^{\beta t}$$

$$S(f) = \beta f$$
$$\phi(t) = \frac{\omega_{st}}{\beta}(e^{\beta t} - 1)$$

where ω_{st} is the start frequency.
In situations where a log frequency axis is desired, Δf usually increase with frequency and the condition:

$$\frac{S}{\Delta f} < f_{cut}$$

or

$$\frac{\beta f}{\Delta f} < f_{cut}$$

can be fulfilled even though it is required that $f_{cut} < 2f_{st}$ for a tracking filter having a fixed cut-off frequency. The tracking filter is in this situation usually implemented similar to the one used for linear sweep.
If the main purpose of using an exponential sweep is to improve the S/N ratio at lower frequencies and a linear frequency axis is desired, then a tracking filter with a cut-off frequency increasing proportionally with frequency can be used, including a conversion from log frequency axis to linear frequency axis.
Since S is proportional to the frequency (βf), the tracking filter bandwidth must also be proportional to the frequency.
The frequency difference df between two consecutive zero crossings is constant and independent of the frequency. This means that if the output of the signal from the mixer is integrated over $\frac{df}{2}$ for each period of the double frequency, then the double frequency is canceled completely and the axis is converted to a linear axis at the same time. An integer number of periods can be used if less resolution is desired. In practice the integration may be combined with a hanning window (or another suitable window).
If the sweep rate is low no further processing is necessary, but for higher sweep rates the frequency response so far obtained is incorrect. Appendix B gives the details of the required processing. It can be summarized as follows:

1. Apply an IFFT to the false frequency response.
2. Stretch the result (the false impulse response) acording to $\frac{1}{b}(1-e^{-tb})$ followed by a damping e^{-tb} in order to obtain the correct impulse response.
3. Apply an FFT to the impulse response in order to obtain the correct frequency response.

Note that the size of the FFT is much smaller than the sweep time due to the sampling reduction.

Example

Figure 2.6 shows the response of a system having three resonances at 1 kHz, 2.5 kHz and 4 kHz (B = 10 Hz). The generator signal is an exponential sweep with $f = 500e^{0.5t}$:
Calculating the false impulse response and analysing it using short time FFT, a contour plot can be obtained showing the effect of the high sweep rate on the impulse response, Figure 2.7.
Note that the contour plot contains three non parallel curved lines. As a consequence the procedure used for linear sweep is not possible, but following the procedure given in Appendix B, the result in Figure 2.8 is obtained.

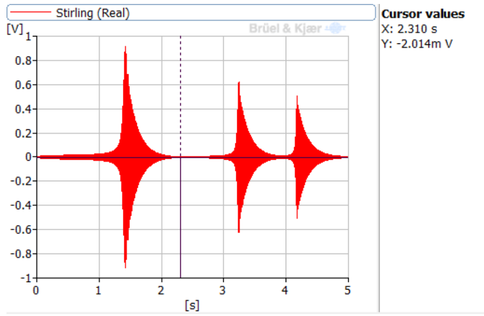

Figure 2.6: Exponential sweep response

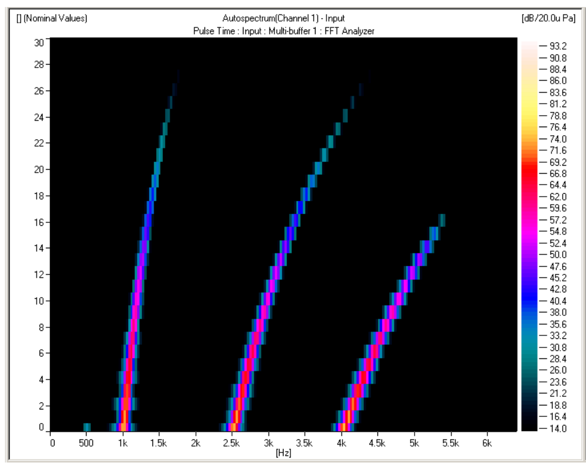

Figure 2.7: Contour plot of the false impulse response

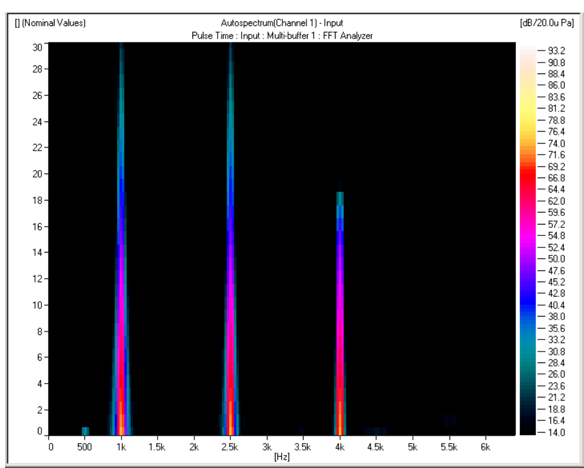

Figure 2.8: Contour plot of the corrected impulse response

Chapter 3

Appendix A

The response of a linear time invariant system is given by:

$$y = h * x$$

where:
x is the excitation signal.
h is the impulse response of the system.

If x is a complex linear sweep the relation can be written as:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{j\phi(t-\tau)}d\tau$$

where:

$$\phi(t) = \pi St^2$$

the output response can now be written:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{j\pi S(t-\tau)^2}d\tau$$

If the output is multiplied by the complex conjugate of the sweep then:

$$z(t) = e^{-j\pi St^2}\int_{-\infty}^{\infty} h(\tau)e^{j\pi S(t-\tau)^2}d\tau = \int_{-\infty}^{\infty} e^{j\pi S\tau^2}h(\tau)e^{-j2\pi St\tau}d\tau =$$

or

$$z(f) = \int_{-\infty}^{\infty} e^{j\pi S\tau^2}h(\tau)e^{-j2\pi f\tau}d\tau = \int_{-\infty}^{\infty} g(s)e^{-j2\pi fs}d\tau$$

For a physical system it is not possible to use a complex excitation signal, but if a real valued sweep is used the result above can be obtained by using a tracking filter for removing the mirror frequency.
It is seen that $g(s)$ can be obtained by an inverse Fourier transform of $z(f)$ and $h(t)$ can then be derived from $g(s)$:

$$g(s) = e^{j\pi Ss^2}h(s)$$

and finally:

$$h(t) = e^{-j\pi St^2}g(t)$$

Chapter 4

Appendix B

The response of a linear time invariant system is given by:

$$y = h * x$$

where:
x is the excitation signal.
h is the impulse response of the system.

If x is a complex exponential sweep the relation can be written as:

$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{j\phi(t-\tau)} d\tau$$

where:

$$\phi(t) = \frac{2\pi f_{st}}{\beta} e^{\beta t}$$

Using the substitutions:

$$\begin{aligned} t &= \frac{1}{\beta} \log(\frac{f}{f_{st}}) \\ \tau &= -\frac{1}{\beta} \log(1 - \beta s) \\ d\tau &= \frac{1}{1 - \beta s} ds \end{aligned}$$

the output response can be written:

$$y(\frac{1}{\beta} ln(\frac{f}{f_{st}})) = \int_{-\infty}^{\frac{1}{\beta}} \frac{1}{1 - \beta s} h(-\frac{1}{\beta} ln(1 - \beta s)) e^{j2\pi f(\frac{1}{\beta} - s)} ds$$

If the output is multiplied by the complex conjugate of the sweep one obtains:

$$z(f) = e^{-j\frac{2\pi}{\beta} f} \int_{-\infty}^{\frac{1}{\beta}} \frac{1}{1 - \beta s} h(-\frac{1}{\beta} ln(1 - \beta s)) e^{j2\pi f(\frac{1}{\beta} - s)} ds$$

or

$$z(f) = \int_{-\infty}^{\frac{1}{\beta}} \frac{1}{1-\beta s} h(-\frac{1}{\beta} ln(1-\beta s)) e^{-j2\pi f s} ds = \int_{-\infty}^{\frac{1}{\beta}} g(s) e^{-j2\pi f s} ds$$

For a physical system it is not possible to use a complex excitation signal, but if a real valued sweep is used the result above can be obtained by using a tracking filter for removing the mirror frequency. It is seen that $g(s)$ can be obtained by an inverse Fourier transform of $z(f)$ ($g(s) = 0$ for $\frac{1}{\beta} < s < \infty$) and $h(t)$ can then be derived from $g(t)$:

$$g(s) = \frac{1}{1-\beta s} h(-\frac{1}{\beta} ln(1-\beta s))$$

and finally:

$$h(t) = e^{-t\beta} g(\frac{1}{\beta}(1-e^{-t\beta}))$$