Generating sine waves using a buffer of limited size

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Sine waves can often be generated in real time using a recursion formula or a table lookup, but it requires a DSP an some times only a buffer that can be played back repeatedly is available. If the buffer length can be varied but is limited to a maximal length N, the frequencies that can be generated are given by:

$$f = \frac{P}{L} f_s, \qquad 2 \leq L \leq N, \qquad 1 \leq P \leq L/2$$

The number of reduced fractions P/L gives the number of distinct frequencies. The set of reduced fractions with a denominator less then N are called Farey fractions. The number of Farey fractions is given by:

$$k = \Sigma\phi(N) \approx \frac{3}{\pi^2} N$$

$\phi(N)$ is Euler's totient function.
For N = 7 the Farey fractions become:

$$\frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}$$

Since P above is limited to L/2 (only frequencies up to half the sampling frequency can be generated) the number of possible frequencies is k/2.
The frequencies are not evenly spaced. It is easily seen that the lowest possible frequency (apart from DC) is $\frac{1}{N} fs$, whereas the average spacing is $\approx \frac{1}{N^2} fs$.
A simple algorithm exists for finding a rational approximation to a number, the denominator being less than N.
Let ξ be the number we whish to aproximate, $x = [\xi]$ and N the maximal denominator:

1. p ← x, q ← 1 , t ← x+1, u ← 1
2. r ← p+t, s ← q+u
3. if s > N, goto 6
4. if p/q ≤ ξ < r/s, then t ← r, u ← s, else p ← r, q ← s
5. goto 2
6. output p,q,t,u

Either p/q or t/u is the best approximation to ξ with a maximal denominator N

Example:

$$\xi = 0.357925, N = 100$$

$\frac{0}{1}$	$\frac{0}{1}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{5}{14}$	$\frac{5}{14}$	$\frac{5}{14}$	$\frac{5}{14}$	$\frac{5}{14}$	$\frac{5}{14}$	$\frac{34}{95}$
$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{3}{8}$	$\frac{4}{11}$	$\frac{4}{11}$	$\frac{9}{25}$	$\frac{14}{39}$	$\frac{19}{53}$	$\frac{24}{67}$	$\frac{29}{81}$	$\frac{29}{81}$

The best approximation is thus 34/95 giving an error of 0.00003. A simple rounding would give 36/100 and an error of 0.002.